

SIMPLICES IN NEWTON-OKOUNKOV BODIES AND THE GROMOV WIDTH OF COADJOINT ORBITS

XIN FANG, PETER LITTELMANN, MILENA PABINIAK

ABSTRACT. We give a uniform proof for the conjectured Gromov width of coadjoint orbits of all compact connected simple Lie groups, by analyzing simplices in Newton-Okounkov bodies.

1. INTRODUCTION

Let ω_{st} be the standard symplectic form on \mathbb{R}^{2n} . The non-squeezing theorem of Gromov affirms that a ball $B^{2n}(r) \subset (\mathbb{R}^{2n}, \omega_{st})$ cannot be symplectically embedded into $B^2(R) \times \mathbb{R}^{2n-2} \subset (\mathbb{R}^{2n}, \omega_{st})$ unless $r \leq R$. This result motivated the quest for the largest ball that could be symplectically embedded into a given symplectic manifold (M, ω) . The *Gromov width* of a $2n$ -dimensional symplectic manifold (M, ω) is the supremum of the set of a 's such that the ball of *capacity* a (radius $\sqrt{\frac{a}{\pi}}$),

$$B_a^{2n} = \left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \pi \sum_{i=1}^n (x_i^2 + y_i^2) < a \right\} \subset (\mathbb{R}^{2n}, \omega_{st}),$$

can be symplectically embedded in (M, ω) .

In this article, we analyze the Gromov width of an important class of symplectic manifolds formed by the orbits of the coadjoint action of compact Lie groups. Let K be a compact Lie group, and let \mathfrak{k}^* be the dual of its Lie algebra \mathfrak{k} . Each orbit $\mathcal{O} \subset \mathfrak{k}^*$ of the coadjoint action of K on \mathfrak{k}^* is naturally equipped with the Kostant-Kirillov-Souriau symplectic form, ω^{KKS} , defined by:

$$\omega_\xi^{KKS}(X^\#, Y^\#) = \langle \xi, [X, Y] \rangle, \quad \xi \in \mathcal{O} \subset \mathfrak{k}^*, \quad X, Y \in \mathfrak{k},$$

where $X^\#, Y^\#$ are the vector fields on \mathfrak{k}^* induced by $X, Y \in \mathfrak{k}$ via the coadjoint action of K . It has been conjectured that the Gromov width of a coadjoint orbit $(\mathcal{O}_\lambda, \omega^{KKS})$ of K , through a point λ in a positive Weyl chamber, is given by the formula (1.1). Although the conjecture had been proved in many cases (see Section 2), it was unsatisfactory that the proofs for the lower bounds were different for each group. As the conjectured Gromov width can be expressed by one formula for all compact connected Lie groups, (1.1), one would like to have a uniform proof which works for all these Lie groups.

The current article provides such a proof. Our main theorem is the following:

Theorem 1.1. *Let K be a compact connected simple Lie group. The Gromov width of a coadjoint orbit \mathcal{O}_λ through λ , equipped with the Kostant-Kirillov-Souriau symplectic form, is at least*

$$(1.1) \quad \min\{|\langle \lambda, \alpha^\vee \rangle|; \alpha^\vee \text{ a coroot and } \langle \lambda, \alpha^\vee \rangle \neq 0\}.$$

Combining this result with the upper bounds proved in [5], we immediately obtain:

Corollary 1.2. *Let K be a compact connected simple Lie group. The Gromov width of a coadjoint orbit \mathcal{O}_λ through λ , equipped with the Kostant-Kirillov-Souriau symplectic form, is given by (1.1).*

The paper is organized as follows: After some comments (Section 2) about the history of the subject and the development of the mathematical tools related to the problem, we recall (Section 3) in more detail how Newton-Okounkov bodies can be used to analyze the Gromov width. In Section 4, we give a proof of Theorem 1.1 up to a construction of certain simplices embedded in Newton-Okounkov bodies for coadjoint orbits \mathcal{O}_λ associated to integral weights. This is done in Section 6, after recalling Lie-theoretic constructions of Newton-Okounkov bodies (Section 5). In Section 7.1 and 7.2 we provide two alternative constructions of simplices embedded in Newton-Okounkov bodies.

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2. A BIT OF HISTORY AND METHODS FOR FINDING LOWER BOUNDS OF THE GROMOV WIDTH

Recall that every coadjoint orbit intersects a chosen positive Weyl chamber in a single point, providing a bijection between the coadjoint orbits and points in a positive Weyl chamber. Orbits intersecting the interior of a positive Weyl chamber are called *generic orbits*. They are of maximal dimension among coadjoint orbits of K , and are diffeomorphic to the quotient K/S , where S is a maximal torus of K . Orbits intersecting a positive Weyl chamber at its boundary are called *degenerate orbits*. For example, when $K = U(n, \mathbb{C})$ is the unitary group, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. The orbit is generic if all eigenvalues are different, and in this case it is diffeomorphic to the manifold of complete flags in \mathbb{C}^n .

Many cases of the conjecture about the Gromov width of coadjoint orbits had already been proved:

- Karshon and Tolman in [13], and independently Lu in [18], proved the conjecture for complex Grassmannians (which are degenerate coadjoint orbits of $U(n, \mathbb{C})$);
- Zoghi in [28] proved it for generic indecomposable orbits of $U(n, \mathbb{C})$;
- Moreover, he showed that the formula (1.1) gives an upper bound for the Gromov width of generic, indecomposable orbits of any compact connected Lie group;
- In [5], Caviedes Castro extended the above result about the upper bound by removing the generic and indecomposable assumptions. This concludes the proof of the upper bound part of the conjecture;
- The third author showed in [25] that the formula (1.1) gives a lower bound of the Gromov width of (not necessarily generic) coadjoint orbits of $U(n, \mathbb{C})$, $SO(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$. (The result about $SO(2n+1, \mathbb{C})$ works only for orbits satisfying one mild technical condition; see [25] for more details);
- Halacheva and the third author in [11] proved that the lower bound is given by (1.1) for generic orbits of the symplectic group $\mathrm{Sp}(n) = U(n, \mathbb{H})$;
- Lane in [16] proved that the lower bound is given by (1.1) for generic orbits of the exceptional group G_2 ;

- Additionally, some of the coadjoint orbits fall into the category of manifolds analyzed by Loi and Zuddas, and their Gromov widths are found in [21].

We briefly explain how one may prove claims like Theorem 1.1. Lower bounds of the Gromov width are found by providing explicit symplectic embeddings of balls. If the given manifold M is equipped with a Hamiltonian (thus effective) action of a compact torus S , such embeddings can be constructed by “flowing along” the flow of the vector fields induced by the action ([13, Proposition 2.8]) and can be read off from the image of the momentum map¹ $\Phi: M \rightarrow \mathfrak{s}^*$ associated to this Hamiltonian action. The situation is especially nice if the action is *toric*, that is, the dimension of the torus is equal to the complex dimension of the manifold. We describe this basic case more carefully. Identify \mathfrak{s}^* with $\mathbb{R}^{\dim S}$, thinking of the circle as $S^1 = \mathbb{R}/\mathbb{Z}$, so that the lattice of \mathfrak{s}^* is mapped to $\mathbb{Z}^{\dim S}$ in $\mathbb{R}^{\dim S}$. The momentum map for the standard $S = (S^1)^n$ action on $(\mathbb{R}^{2n}, \omega_{st})$ maps a ball of capacity a into an n -dimensional simplex of size a , closed on n sides:

$$(2.1) \quad \mathfrak{S}^n(a) := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_j < a, \sum_{j=1}^n x_j < a\}.$$

Conversely, if for a manifold (M^{2n}, ω) of dimension $2n$, equipped with a toric action of an n -dimensional torus S , and the momentum map $\Phi: M \rightarrow \mathfrak{s}^* \cong \mathbb{R}^n$, there exist $\Psi \in GL(n, \mathbb{Z})$, $x \in \mathbb{R}^n$, and an open half-space $H := \{y \in \mathbb{R}^n \mid \langle y, v \rangle > a\}$ such that

$$\Psi(\mathfrak{S}^n(a)) + x = \Phi(M) \cap H,$$

then a ball of capacity a can be symplectically embedded in (M^{2n}, ω) . The appearance of $\Psi \in GL(n, \mathbb{Z})$ arises from the non-canonical identification of \mathfrak{s}^* with \mathbb{R}^n (it depends on a chosen splitting of S into a product of circles), whereas x appears because the momentum map is unique only up to adding a constant. This result had been later generalized to open simplices contained somewhere in the momentum map image (*i.e.*, not necessarily being $\Phi(M) \cap H$). Then one obtains only embeddings of balls of capacities $a - \varepsilon$ for any $\varepsilon > 0$, still implying that the Gromov width is at least a . More precisely:

Proposition 2.1. [19, Proposition 1.3][25, Proposition 2.5] *For any connected, proper (not necessarily compact) Hamiltonian $(S^1)^n$ -space M of dimension $2n$, with a momentum map Φ , the Gromov width of M is at least*

$$\sup\{a > 0 \mid \exists \Psi \in GL(n, \mathbb{Z}), x \in \mathbb{R}^n, \text{ such that } \Psi(\text{int } \mathfrak{S}^n(a)) + x \subset \Phi(M)\}.$$

In the case where the action is not toric, one needs to look for projections of simplices not in the whole moment map image, but in its part called a centered region ([13, Proposition 2.8]). Coadjoint orbits of K are equipped with Hamiltonian (though usually not toric) actions of the maximal torus of K . Applying [13, Proposition 2.8] to this action, Zoghi in [28] constructed symplectic embeddings, proving that the Gromov widths of generic indecomposable coadjoint orbits of $U(n)$ are given by (1.1). For non-simply laced groups, the same trick does not give the expected lower bound, but a weaker one ([26, Appendix A]). To obtain the good lower bounds for the coadjoint orbits of $SO(2n)$ and $SO(2n+1)$, (*i.e.*, equal to (1.1)), a different action was used: a Gelfand-Tsetlin action ([25]). This action is defined

¹The momentum map for a Hamiltonian S action on M is a map $\Phi: M \rightarrow \mathfrak{s}^*$ such that for any $\xi \in \mathfrak{s}$ the differential of $p \mapsto \Phi(p)(\xi)$ is equal to $\omega(\xi, \cdot)$. Thus it is unique only up to adding a constant.

only on an open dense subset of the orbit, but there it is toric and therefore provides embeddings of relatively big balls. This approach fails for the symplectic group. The corresponding Gelfand-Tsetlin action is not toric in that case, and although one still obtains some symplectic embeddings of balls, these balls are of capacities smaller than the expected Gromov width.

A new upgrade in these tools came with the work of Harada and Kaveh [12], where the idea of a toric degeneration was brought from algebraic to symplectic geometry. Toric degeneration of a complex algebraic variety X is a flat family over \mathbb{C} , with generic fibers X_z isomorphic to X , and the special fiber X_0 being a toric variety. One constructs such a degeneration from a very ample Hermitian line bundle over X and a valuation on its sections (satisfying certain assumptions). Harada and Kaveh showed how to create a degeneration of a given symplectic manifold (with some relatively mild assumptions) to a toric variety, keeping track of the symplectic form, and in such a way that the toric action on that variety can be pulled back to a toric action (so Hamiltonian with respect to the given symplectic structure) on an open dense subset of the symplectic manifold (Theorem 3.2). A toric action obtained in this way was used by Halacheva and the third author in [11] to prove that the Gromov width of the coadjoint orbits of the symplectic group $\mathrm{Sp}(n) = U(n, \mathbb{H})$ has a lower bound as in (1.1). As this type of argument could be used for any (compact connected and simple) Lie group, it prompted the idea of having one unified proof for all coadjoint orbits. In the next section, we explain in more detail how this new tool can be used to analyze the Gromov width.

3. NEWTON-OKOUNKOV BODIES AND ROOT SUBGROUPS

Throughout the article, \mathfrak{g} is the Lie algebra of a simply connected simple complex algebraic group G of rank n . We fix a Borel subalgebra \mathfrak{b} of \mathfrak{g} containing a fixed Cartan subalgebra \mathfrak{t} . Let $B \subset G$ be the corresponding Borel subgroup, $T \subset B$ be the maximal torus and U^- be the unipotent radical of B^- , opposite to B . Let λ be a dominant integral weight and denote by $\mathrm{supp}(\lambda)$ the *support* of λ , i.e., the set of fundamental weights occurring with a nonzero coefficient in writing λ into a sum of fundamental weights. The set of dominant integral weights is denoted by Λ^+ . For the irreducible representation $V(\lambda)$ of G of highest weight λ , let $\mathbb{C}v_\lambda$ be the highest weight line. Let $P = P_\lambda \supseteq B$ be the normalizer in G of this line. Recall that P depends only on the support $\mathrm{supp}(\lambda)$ of λ , not on the weight itself. The associated line bundle \mathcal{L}_λ on G/P is very ample, thus, after fixing a Hermitian structure on \mathcal{L}_λ , one can equip G/P with a symplectic structure ω_λ induced from the Fubini-Study form on the projective space $\mathbb{P}(H^0(G/P, \mathcal{L}_\lambda)^*) = \mathbb{P}(V(\lambda))$ via the Kodaira embedding.

Let $K \subset G$ be a maximal compact subgroup such that $S = T \cap K$ is a maximal torus in K , and let $K_P = K \cap P$. Then $G/P = K/K_P$ can be identified with the highest weight orbit $G \cdot [v_\lambda] \subset \mathbb{P}(V(\lambda))$. Recall that a dominant integral weight is an integral point in the chosen positive Weyl chamber. The coadjoint orbit \mathcal{O}_λ of K through λ , is diffeomorphic to K/K_P , and, when equipped with the Kostant-Kirillov-Souriau symplectic form, it is symplectomorphic to $(G/P, \omega_\lambda)$ (see for example [5, Remark 5.5]).

To construct toric degenerations of G/P , we use the theory of Newton-Okounkov bodies [15, 20] and the method of birational sequences [7]. Let U_P^- be the unipotent

radical of the parabolic subgroup P^- , opposite to P . The birational map $U_P^- \rightarrow G/P$, $u \mapsto u \cdot [\text{id}]$, induces an isomorphism of fields $\mathbb{C}(U_P^-) \simeq \mathbb{C}(G/P)$.

For a positive root β , denote by $\mathfrak{g}_{-\beta} \subset \text{Lie } U^-$ the associated root subspace and let $U_{-\beta} = \exp \mathfrak{g}_{-\beta} \subset U^-$ be the corresponding root subgroup. Denote by Φ_P^+ the set of positive roots $\beta \in \Phi^+$ such that $\mathfrak{g}_{-\beta} \subset \text{Lie } U_P^-$. We fix an enumeration $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ of the roots in Φ_P^+ . The product map

$$\pi : U_{-\beta_1} \times \dots \times U_{-\beta_N} \rightarrow U_P^-$$

is known to be an isomorphism of affine varieties. We write $\mathbb{C}[x_{\beta_i}]$ for the coordinate ring of $U_{-\beta_i}$, which is, as affine variety, just an affine line.

Let $>_r$ be the right lexicographic order on \mathbb{N}^N : for two tuples $\underline{\mathbf{m}} = (m_1, m_2, \dots, m_N)$ and $\underline{\mathbf{k}} = (k_1, k_2, \dots, k_N)$, we say that $\underline{\mathbf{m}} >_r \underline{\mathbf{k}}$, if there exists $1 \leq s \leq N$ such that $m_N = k_N, \dots, m_{s+1} = k_{s+1}$, and $m_s > k_s$. We get an induced monomial order on the set of monomials in $\mathbb{C}[x_{\beta_1}, \dots, x_{\beta_N}]$ by defining $x^{\underline{\mathbf{m}}} >_r x^{\underline{\mathbf{k}}}$ if $\underline{\mathbf{m}} >_r \underline{\mathbf{k}}$, where for $\underline{\mathbf{t}} = (t_1, \dots, t_N)$, $x^{\underline{\mathbf{t}}} := x_{\beta_1}^{t_1} \dots x_{\beta_N}^{t_N}$. We define an induced \mathbb{Z}^N -valued valuation on $\mathbb{C}(G/P) = \mathbb{C}(x_{\beta_1}, \dots, x_{\beta_N})$ in the following way: for a nonzero polynomial in $\mathbb{C}[x_{\beta_1}, \dots, x_{\beta_N}]$,

$$\nu(\sum a_{\underline{\mathbf{m}}} x^{\underline{\mathbf{m}}}) := \min\{\underline{\mathbf{m}} \mid a_{\underline{\mathbf{m}}} \neq 0\}$$

and $\nu(\frac{f}{g}) = \nu(f) - \nu(g)$ for a nonzero element $f/g \in \mathbb{C}(G/P)$.

Let R_λ be the ring of sections

$$R_\lambda = \bigoplus_{\ell \geq 0} H^0(G/P, \mathcal{L}_\lambda^{\otimes \ell}).$$

We fix a nonzero highest weight section $s_0 \in H^0(G/P, \mathcal{L}_\lambda)$ and form a graded monoid (*i.e.*, a graded semigroup with identity)

$$\Gamma_{\lambda, \underline{\beta}} = \bigcup_{\ell \in \mathbb{N}} \Gamma_{\lambda, \underline{\beta}}(\ell) \subset \mathbb{N} \times \mathbb{Z}^N, \quad \text{where } \Gamma_{\lambda, \underline{\beta}}(\ell) = \{(\ell, \nu(s/s_0^\ell)) \mid s \in H^0(G/P, \mathcal{L}_\lambda^{\otimes \ell})\}.$$

The Newton-Okounkov body $\Delta_\lambda(\underline{\beta})$ associated to the valuation is

$$\Delta_\lambda(\underline{\beta}) = \overline{\left\{ \frac{1}{\ell} \underline{\mathbf{m}} \mid (\ell, \underline{\mathbf{m}}) \in \Gamma_{\lambda, \underline{\beta}} \right\}} \subset \mathbb{R}^N.$$

The following theorems are results of Anderson [1], Harada and Kaveh [12]. They hold in a much more general situation, but here we rephrase them according to the special circumstance of this article.

Theorem 3.1 ([1]). *If the monoid $\Gamma_{\lambda, \underline{\beta}}$ is finitely generated, there exists a flat family $\pi : \mathfrak{X} \rightarrow \mathbb{C}$ such that for any $z \in \mathbb{C} \setminus \{0\}$, the fibre $X_z = \pi^{-1}(z)$ is isomorphic to G/P , and $X_0 = \pi^{-1}(0)$ is isomorphic to $\text{Proj } \mathbb{C}[\Gamma_{\lambda, \underline{\beta}}]$. The variety X_0 is equipped with an action of the torus $(\mathbb{C}^*)^N$. The normalization of the variety X_0 is the toric variety associated to the rational polytope $\Delta_\lambda(\underline{\beta})$.*

Moreover, the torus action on X_0 induces a torus action on (a subset of) G/P .

Theorem 3.2 ([12]). *Assume that $\Gamma_{\lambda, \underline{\beta}}$ is finitely generated. There exists an integrable system $\mu = (\mathcal{F}_1, \dots, \mathcal{F}_N) : G/P \rightarrow \mathbb{R}^N$ on $(G/P, \omega_\lambda)$, and the image of μ coincides with the Newton-Okounkov body $\Delta_\lambda(\underline{\beta})$. The integrable system generates a torus action on the inverse image under μ of the interior of $\Delta_\lambda(\underline{\beta})$, and there the restriction of μ is a momentum map.*

Remark 3.3. *In fact, it follows from the proof of Theorem 2.19, together with Corollary 2.11 and Theorem 2.12 from [12], that the torus action is defined on the preimage of the smooth² points of $\Delta_\lambda(\underline{\beta})$. Therefore, if there exist $\Psi \in GL(n, \mathbb{Z})$, $x \in \mathbb{R}^n$, and an open half-space $H := \{y \in \mathbb{R}^n \mid \langle y, v \rangle > a\}$ such that*

$$\Psi(\mathfrak{S}^n(a)) + x = \Delta_\lambda(\underline{\beta}) \cap H,$$

then a ball of capacity a can be symplectically embedded in $(G/P, \omega_\lambda)$.

To show the finite generation of the monoid $\Gamma_{\lambda, \underline{\beta}}$ is usually (not only in the flag variety case) a difficult problem. Some special cases are presented in Section 5.

The Newton-Okounkov body may provide interesting information even if $\Gamma_{\lambda, \underline{\beta}}$ is *not necessarily finitely generated*. The following result is due to Kaveh [14, Corollary 12.3 and 12.4]. Again, we rephrase it for the special case where $X = G/P$ is a flag variety. Recall that by $\text{int } \mathfrak{S}^N(r)$ we denote the interior of the N -dimensional simplex of size r .

Theorem 3.4 ([14]). *The Gromov width of $(G/P, \omega_\lambda)$ is at least R , where R is the supremum of the sizes of open simplices that fit (up to $GL(N, \mathbb{Z})$ transformation) in the interior of the Newton-Okounkov body $\Delta_\lambda(\underline{\beta})$, i.e.,*

$$R = \sup\{r > 0 \mid \exists \Psi \in GL(N, \mathbb{Z}), x \in \mathbb{R}^N, \text{ such that } \Psi(\text{int } \mathfrak{S}^N(r)) + x \subset \Delta_\lambda(\underline{\beta})\}.$$

4. THE PROOF OF THE MAIN RESULT.

With this background reviewed, we are now ready to present the proof of Theorem 1.1, up to the detailed analysis of the Newton-Okounkov bodies, which is postponed to the next sections.

Proof of Theorem 1.1. We first prove Theorem 1.1 for integral λ in a positive Weyl chamber, i.e., for a dominant integral weight. Then the coadjoint orbit $(\mathcal{O}_\lambda, \omega^{KKS})$ is symplectomorphic to a flag manifold $(G/P, \omega_\lambda)$ with a symplectic structure pulled back via Kodaira embedding $G/P \hookrightarrow \mathbb{P}(H^0(G/P, \mathcal{L}_\lambda)^*)$.

In the previous section, we explained how to associate a valuation on the space of sections of \mathcal{L}_λ (and of its tensor products) to a given enumeration $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ of roots in Φ_P^+ . From there one obtains a Newton-Okounkov body $\Delta_\lambda(\underline{\beta})$. In Section 6 we fix an enumeration $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ of the positive roots in question and analyze the associated Newton-Okounkov bodies $\Delta_\lambda(\underline{\beta})$. We show (see Theorem 6.2) that the body contains an open simplex of the size precisely as in (1.1). Then Theorem 5.5 implies Theorem 1.1 (and thus also Corollary 1.2) for an integral λ . (See also Section 7.1 and 7.2 for different enumerations).

If λ is rational, take ℓ such that $\ell\lambda$ is integral. Then $(\mathcal{O}_\lambda, \ell\omega^{KKS})$ is symplectomorphic to $(\mathcal{O}_{\ell\lambda}, \omega^{KKS})$ and thus the Gromov width of $(\mathcal{O}_\lambda, \omega^{KKS})$ is $\frac{1}{\ell}$ of the Gromov width of $(\mathcal{O}_{\ell\lambda}, \omega^{KKS})$, i.e.,

$$\begin{aligned} & \frac{1}{\ell} \min\{|\langle \alpha^\vee, \ell\lambda \rangle|; \alpha^\vee \text{ a coroot and } \langle \alpha^\vee, \lambda \rangle \neq 0\} \\ &= \min\{|\langle \alpha^\vee, \lambda \rangle|; \alpha^\vee \text{ a coroot and } \langle \alpha^\vee, \lambda \rangle \neq 0\}. \end{aligned}$$

² Recall that a polytope $\Delta \in \mathbb{R}^n$ is called *smooth* if there are exactly n edges meeting at each vertex of Δ and their primitive generators form a \mathbb{Z} basis of \mathbb{Z}^n (see for example [6, Definition 2.4.2]). A point x in the interior of a facet F of a polytope Δ is called *smooth*, if F itself is a smooth polytope. Note that for a simplex all points are smooth.

Extending to irrational λ is done by a “Moser type” argument, described in detail in [11] (see also [22]). We point out that in this “Moser type” argument one needs to know the actual Gromov widths of orbits through rational λ , not just the lower bounds. \square

5. ESSENTIAL MONOMIALS

Let λ be a dominant integral weight and let $P = P_\lambda$ be the associated parabolic subgroup. In [7], the first two authors give a representation-theoretic construction of the monoid $\Gamma_{\lambda, \underline{\beta}}$ coming from an enumeration $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ of roots in Φ_P^+ . For the rest of this section, we fix the enumeration $\underline{\beta}$ and remove it from the notations for simplicity.

5.1. Filtration arising from birational sequences. Let \mathfrak{n}_P^- be the Lie algebra of U_P^- . For each positive root β , let F_β be a generator of $\mathfrak{g}_{-\beta}$, E_β be a generator of \mathfrak{g}_β such that the sub-algebra generated by E_β , F_β and β^\vee is isomorphic to \mathfrak{sl}_2 . The vectors $\{F_\beta \mid \beta \in \Phi_P^+\}$ form a vector space basis of \mathfrak{n}_P^- . As a vector space, the enveloping algebra $U(\mathfrak{n}_P^-)$ admits a PBW-basis (*i.e.*, ordered monomials in the root vectors):

$$\{F^{\underline{\mathbf{m}}} = F_{\beta_1}^{m_1} \cdots F_{\beta_N}^{m_N} \mid \underline{\mathbf{m}} = (m_1, m_2, \dots, m_N) \in \mathbb{N}^N\}.$$

Let $>_{\text{or}}$ be the right opposite lexicographic order on \mathbb{N}^N : that is to say, $\underline{\mathbf{m}} >_{\text{or}} \underline{\mathbf{k}}$ if and only if $\underline{\mathbf{m}} <_{\text{r}} \underline{\mathbf{k}}$. We use this total order to define for $\underline{\mathbf{m}} \in \mathbb{N}^N$ subspaces of $U(\mathfrak{n}_P^-)$ as follows:

$$U(\mathfrak{n}_P^-)_{<_{\text{or}} \underline{\mathbf{m}}} = \text{span}_{\mathbb{C}}\{F^{\underline{\mathbf{k}}} \mid \underline{\mathbf{k}} <_{\text{or}} \underline{\mathbf{m}}\}, \quad U(\mathfrak{n}_P^-)_{\leq_{\text{or}} \underline{\mathbf{m}}} = \text{span}_{\mathbb{C}}\{F^{\underline{\mathbf{k}}} \mid \underline{\mathbf{k}} \leq_{\text{or}} \underline{\mathbf{m}}\}.$$

For an irreducible representation $V(\lambda)$, we have induced subspaces of $V(\lambda)$:

$$(5.1) \quad V(\lambda)_{<_{\text{or}} \underline{\mathbf{m}}} = (U(\mathfrak{n}_P^-)_{<_{\text{or}} \underline{\mathbf{m}}}) \cdot v_\lambda, \quad V(\lambda)_{\leq_{\text{or}} \underline{\mathbf{m}}} = (U(\mathfrak{n}_P^-)_{\leq_{\text{or}} \underline{\mathbf{m}}}) \cdot v_\lambda.$$

The subquotient $V(\lambda)_{\leq_{\text{or}} \underline{\mathbf{m}}} / V(\lambda)_{<_{\text{or}} \underline{\mathbf{m}}}$ is obviously at most one dimensional.

Definition 5.1. A tuple $\underline{\mathbf{m}} \in \mathbb{N}^N$, the monomial $F^{\underline{\mathbf{m}}}$ and the vector $F^{\underline{\mathbf{m}}}v_\lambda$ are called *essential* for $V(\lambda)$, if the dimension of the subquotient $V(\lambda)_{\leq_{\text{or}} \underline{\mathbf{m}}} / V(\lambda)_{<_{\text{or}} \underline{\mathbf{m}}}$ is equal to one.

5.2. Essential monoids and global version. We collect all essential tuples for $V(\lambda)$ in the set $\text{es}_P(\lambda)$. More generally, for $\ell \in \mathbb{N}$, we set

$$\text{es}_P(\ell\lambda) = \{(\ell, \underline{\mathbf{m}}) \in \mathbb{N} \times \mathbb{Z}^N \mid \underline{\mathbf{m}} \text{ is essential for } V(\ell\lambda)\} \subset \mathbb{N} \times \mathbb{Z}^N.$$

It has been shown in [7, Proposition 1] that for integers $\ell, k \geq 1$, $\text{es}_P(\ell\lambda) + \text{es}_P(k\lambda) \subset \text{es}_P((\ell + k)\lambda)$ (here $+$ stands for the Minkowski sum of two sets), therefore the set

$$\text{Es}_P(\lambda) = \bigcup_{\ell \in \mathbb{N}} \text{es}_P(\ell\lambda) \subset \mathbb{N} \times \mathbb{Z}^N$$

is naturally endowed with the structure of a submonoid of $\mathbb{N} \times \mathbb{Z}^N$. Moreover:

Theorem 5.2 ([7]). *The graded submonoids Γ_λ and $\text{Es}_P(\lambda)$ of $\mathbb{N} \times \mathbb{Z}^N$ coincide.*

Let μ be a dominant integral weight such that $\text{supp}(\mu) \subseteq \text{supp}(\lambda)$. In this case $V(\mu)$ is still a cyclic $U(\mathfrak{n}_P^-)$ -module, so the filtration described in (5.1) for $V(\lambda)$ also makes sense for $V(\mu)$, and so does the notation of an essential tuple:

Definition 5.3. A tuple $\underline{\mathbf{m}} \in \mathbb{N}^N$, the monomial $F^{\underline{\mathbf{m}}}$ and the vector $F^{\underline{\mathbf{m}}}v_\mu$ are called *essential* for $V(\mu)$, if the dimension of the subquotient $V(\mu)_{\leq \text{or } \underline{\mathbf{m}}} / V(\mu)_{< \text{or } \underline{\mathbf{m}}}$ is equal to one. Denote the set of all essential tuples for $V(\mu)$ by

$$\text{es}_P(\mu) = \{\underline{\mathbf{m}} \in \mathbb{N}^N \mid \underline{\mathbf{m}} \text{ is essential for } V(\mu)\}$$

and set

$$\text{Es}_P = \{(\mu, \underline{\mathbf{m}}) \in \Lambda^+ \times \mathbb{Z}^N \mid \text{supp}(\mu) \subseteq \text{supp}(\lambda), \underline{\mathbf{m}} \text{ is essential for } V(\mu)\}.$$

Theorem 5.4 ([7]). *The set $\text{Es}_P \subset \Lambda^+ \times \mathbb{Z}^N$ is a submonoid. In particular, for $(\mu, \underline{\mathbf{m}}), (\nu, \underline{\mathbf{k}}) \in \text{Es}_P$, one has $\underline{\mathbf{m}} + \underline{\mathbf{k}} \in \text{es}_P(\mu + \nu)$.*

Consider the following class of examples. Let $\{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots for \mathfrak{g} , $L \supseteq T$ be the Levi subgroup of P and w_L be the longest word in the Weyl group of L . Fix a reduced decomposition of the longest word w_0 in the Weyl group W of G , which is of the form $w_0 = w_L s_{i_1} \cdots s_{i_N}$ or $w_0 = s_{i_1} \cdots s_{i_N} w_L$: in the first case we set

$$(5.2) \quad \Phi_P^+ = \{\beta_1 = w_L(\alpha_{i_1}), \beta_2 = w_L s_{i_1}(\alpha_{i_2}), \dots, \beta_N = w_L s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})\};$$

while in the second case we set

$$(5.3) \quad \Phi_P^+ = \{\beta_N = w_L(\alpha_{i_N}), \beta_{N-1} = w_L s_{i_N}(\alpha_{i_{N-1}}), \dots, \beta_1 = w_L s_{i_N} \cdots s_2(\alpha_{i_1})\}.$$

An enumeration as in (5.2) or (5.3) is said to be *induced by a reduced decomposition*.

Theorem 5.5. [7, Corollary 6] *If the enumeration of the positive roots is induced by a reduced decomposition, then $\Gamma_{\lambda, \underline{\beta}}$ is finitely generated and saturated. In particular, the limit toric variety X_0 (see Theorem 3.1) is normal.*

Remark 5.6. *Even if $\Gamma_{\lambda, \underline{\beta}}$ is finitely generated, it is not an easy task to give an explicit description of the associated Newton-Okounkov body. If the enumeration of the positive roots is induced by a reduced decomposition, using [23], one can show that these bodies are related to string polytopes. The string polytopes are described in [3, 17], and further examples are given in [2] and [7]. Comments regarding geometric properties of the limit toric variety X_0 (see Theorem 3.1) like Fano and Gorenstein can be found in [2].*

The identification of Γ_λ with $\text{Es}_P(\lambda)$ can be used to construct simplices contained in Δ_λ . In the next two sections, we describe three different methods of doing so.

6. A SPECIAL SIMPLEX IN Δ_λ FROM GOOD ORDERINGS

Recall the usual partial order on positive roots: $\beta \succ \gamma$ if $\beta - \gamma$ can be written as a sum of positive roots. In this section, we assume that the enumeration $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ of the roots in Φ_P^+ is a *good ordering* in the sense of [7], i.e., if $\beta_i \succ \beta_j$, then $i > j$. Again, as $\underline{\beta}$ is fixed throughout the section, we suppress it from the notation. For $1 \leq k \leq N$, we denote by \mathbf{e}_k the standard coordinate of \mathbb{R}^N whose k -th entry is 1 and other entries are 0.

Lemma 6.1. *If ϖ is a fundamental weight contained in $\text{supp}(\lambda)$ and $\beta \in \Phi_P^+$ is such that $\langle \varpi, \beta^\vee \rangle \neq 0$, then the root vector F_β is essential for $V(\varpi)$.*

Proof. Let i_0 be such that $\beta = \beta_{i_0}$ in the enumeration of the roots in Φ_P^+ . Let $F^{\mathbf{k}}$ be an element of the PBW-basis of weight β . Then either $\mathbf{k} = \mathbf{e}_{i_0}$, or one has $\beta = \sum_{j=1}^N k_j \beta_j$ with at least two nonzero coefficients. It follows that if $k_j \neq 0$, then $\beta \succ \beta_j$, and hence $\mathbf{k} >_{\text{or}} \mathbf{e}_{i_0}$ in the opposite right lexicographic ordering. In addition, $\langle \varpi, \beta^\vee \rangle \neq 0$ implies $F_\beta v_\lambda \neq 0$, which proves that $F_\beta v_\lambda$ and hence F_β is essential for $V(\varpi)$. \square

Recall the definition of $\mathfrak{S}^N(1)$ in (2.1). Let ρ_P be the sum of all fundamental weights in $\text{supp}(\lambda)$.

Theorem 6.2. *For $k = \min\{|\langle \lambda, \alpha^\vee \rangle| \mid \alpha^\vee \text{ a coroot and } \langle \lambda, \alpha^\vee \rangle \neq 0\}$, one has $\mathfrak{S}^N(k) \subset \Delta_\lambda$.*

Proof. It is easy to see that $k = \min\{|\langle \lambda, \beta^\vee \rangle| \mid \beta \in \Phi_P^+\}$, and it is the maximal integer such that $\lambda = k\rho_P + \nu$, where ν is a dominant integral weight with $\text{supp}(\nu) \subset \text{supp}(\lambda)$.

Let $\beta_i \in \Phi_P^+$. Since $\langle \lambda, \beta_i^\vee \rangle \neq 0$, one has $\langle \varpi, \beta_i^\vee \rangle \neq 0$ for some fundamental weight $\varpi \in \text{supp}(\lambda)$. By Lemma 6.1, \mathbf{e}_i is essential for $V(\varpi)$ for some $\varpi \in \text{supp}(\lambda)$. Since the zero vector is always among the essential tuples for any dominant weight, Theorem 5.4 implies that $\mathbf{e}_1, \dots, \mathbf{e}_N$ are essential for $V(\rho_P)$. The same reasoning implies that $k\mathbf{e}_1, \dots, k\mathbf{e}_N$ are essential for $V(k\rho_P)$, and, again for the same reason, they are essential for $V(\lambda)$. By the convexity of Δ_λ , one has $\mathfrak{S}^N(k) \subset \Delta_\lambda$. \square

Remark 6.3. *If G is of type A_n, C_n, G_2 or D_4 , then it is known, that there exists a good ordering such that the monoid $\Gamma_\lambda = \text{Es}_P(\lambda)$ is finitely generated for any dominant integral weight λ [8, 9, 10]. We conjecture that this holds for any simply connected simple complex algebraic group G .*

Remark 6.4. *Let G be of one of the types above. It has been shown that in these cases, there exists a good ordering such that for any dominant weight the monoid $\Gamma_\lambda = \text{Es}_P(\lambda)$ is finitely generated, saturated and has the Minkowski property, i.e., $\text{es}_P(k\lambda) + \text{es}_P(\ell\lambda) = \text{es}_P((k+\ell)\lambda)$ [8, 9, 10]. In particular, the Newton-Okounkov body Δ_λ is an integral polytope and the limit variety X_0 (Theorem 3.1) is a normal toric variety. In addition, if $\mathcal{L}_\lambda = \mathcal{O}(-K_{G/P})$ is the anticanonical line bundle, then, by [2, Theorem 3.8], X_0 is a Fano variety and the Newton-Okounkov body is reflexive.*

7. ADDITIONAL SPECIAL SIMPLICES IN Δ_λ

Although the first method (Section 6) gives a proof of Theorem 1.1 in full generality, we present two additional approaches for generic orbits. In these approaches we use enumerations of the positive roots $\underline{\beta}$ induced by reduced decompositions of w_0 , so the monoids $\Gamma_{\lambda, \underline{\beta}}$ are finitely generated and saturated. It follows that the limit varieties X_0 (Theorem 3.1) exist and are normal toric varieties. Moreover, in many of these cases, an explicit description of the Newton-Okounkov body is known (see [3, 16], Remark 5.6). Therefore, we hope that these approaches could help to solve other related problems about coadjoint orbits (for example, symplectic packing, formula for potential functions and finding non-displaceable Lagrangians [24], etc.). Additionally, the degeneration discussed in Section 7.2 has the advantage that the simplex of appropriate size is a “chopped off” corner of the Newton-Okounkov body. Therefore, Remark 3.3 implies that there exists an embedding of a ball of capacity given by (1.1), i.e. the supremum appearing in the Gromov width definition is attained. Moreover, the construction of cominuscule telescopes (Section 7.2) may relate to Thimm’s trick [27].

We assume in the following that λ is a *regular dominant integral weight*, i.e., for any simple root α : $\langle \lambda, \alpha^\vee \rangle > 0$. In this case, $\text{supp}(\lambda)$ is the set of all fundamental weight, so $P = B$ and we omit this subscript to simplify the notation, and (1.1) is equal to $\min\{\langle \lambda, \beta^\vee \rangle \mid \beta \in \Phi^+\}$.

7.1. A special simplex in Δ_λ from convex ordering. Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of the longest word and let $\underline{\beta} = \{\beta_1, \dots, \beta_N\}$ be the induced enumeration of all positive roots as in (5.3) with $L = T$.

We define a tuple $\underline{\mathbf{m}}^{\max} = (m_1^{\max}, \dots, m_N^{\max}) \in \mathbb{N}^N$ by descending induction:

- m_N^{\max} is the maximal integer with the property $F_{\beta_N}^{m_N^{\max}} v_\lambda \neq 0$;
- if m_ℓ^{\max} is defined for $\ell = k+1, \dots, N$, then m_k^{\max} is defined as the maximal integer with the property $F_{\beta_k}^{m_k^{\max}} F_{\beta_{k+1}}^{m_{k+1}^{\max}} \cdots F_{\beta_N}^{m_N^{\max}} v_\lambda \neq 0$.

The tuple $\underline{\mathbf{m}}^{\max}$ gives rise to a sequence of tuples in \mathbb{N}^N : for $k = 1, \dots, N$, set

$$\underline{\mathbf{m}}_k^{\max} = (0, \dots, 0, m_k^{\max}, m_{k+1}^{\max}, \dots, m_N^{\max}).$$

Lemma 7.1. *For all $k = 1, \dots, N$, the following statements hold:*

- (1) $m_k^{\max} = \langle \lambda, \alpha_{i_k}^\vee \rangle > 0$ for all $k = 1, \dots, N$;
- (2) $F_{\beta_k}^{m_k^{\max}} \cdots F_{\beta_N}^{m_N^{\max}} v_\lambda$ is a weight vector of weight $s_{\beta_k} \cdots s_{\beta_N}(\lambda)$;
- (3) $\underline{\mathbf{m}}_k^{\max}$ is essential for $V(\lambda)$.

Proof. The proofs of (1) and (2) are executed by descending induction. Note that for $k = N$, β_N is a simple root. Since λ is a regular dominant weight, \mathfrak{sl}_2 -theory implies (1) and (2).

Suppose now $k < N$ and the claims hold for $k+1$. Then $F_{\beta_{k+1}}^{m_{k+1}^{\max}} \cdots F_{\beta_N}^{m_N^{\max}} v_\lambda$ is an extremal weight vector, hence is either a highest or a lowest weight vector for the subalgebra $\mathfrak{sl}_2(\beta_k)$ of \mathfrak{g} generated by the root vectors F_{β_k}, E_{β_k} and β_k^\vee . Recall from (5.3) that $\beta_k = s_{i_N} \cdots s_{i_{k+1}}(\alpha_{i_k})$. Since

$$\begin{aligned} \langle s_{\beta_{k+1}} \cdots s_{\beta_N}(\lambda), \beta_k^\vee \rangle &= \langle s_{i_N} \cdots s_{i_{k+1}}(\lambda), s_{i_N} \cdots s_{i_{k+1}}(\alpha_{i_k}^\vee) \rangle \\ &= \langle \lambda, \alpha_{i_k}^\vee \rangle \end{aligned}$$

is strictly positive, $F_{\beta_{k+1}}^{m_{k+1}^{\max}} \cdots F_{\beta_N}^{m_N^{\max}} v_\lambda$ is a highest weight vector for $\mathfrak{sl}_2(\beta_k)$. It follows that $m_k^{\max} = \langle \lambda, \alpha_{i_k}^\vee \rangle > 0$, and $F_{\beta_k}^{m_k^{\max}} \cdots F_{\beta_N}^{m_N^{\max}} v_\lambda$ is a weight vector of weight $s_{\beta_k} \cdots s_{\beta_N}(\lambda)$, which proves the claim by induction.

To prove (3), notice that for a tuple $\underline{\mathbf{m}} <_{or} \underline{\mathbf{m}}_k^{\max}$ one has only two possibilities: either $F^{\underline{\mathbf{m}}} v_\lambda = 0$, or the weight of $F^{\underline{\mathbf{m}}} v_\lambda$ is different from the weight of $F^{\underline{\mathbf{m}}_k^{\max}} v_\lambda$. This implies that $F^{\underline{\mathbf{m}}_k^{\max}} v_\lambda$ is linearly independent of the $F^{\underline{\mathbf{m}}} v_\lambda$ with $\underline{\mathbf{m}} <_{or} \underline{\mathbf{m}}_k^{\max}$, and hence $\underline{\mathbf{m}}_k^{\max}$ is essential for $V(\lambda)$. \square

Denote by $\mathbf{e}_{i,N} \in \mathbb{R}^N$ the element $\mathbf{e}_{i,N} = \sum_{j=i}^N \mathbf{e}_j \in \mathbb{R}^N$. Recall that ρ is the sum of all fundamental weights. Lemma 7.1 implies that the tuples $\mathbf{e}_{i,N}$ are elements of $\text{es}(\rho) \subset \Delta_\rho$. Set

$$\mathfrak{S}_\rho^N = \text{Convex hull of } \{0, \mathbf{e}_{1,N}, \mathbf{e}_{2,N}, \dots, \mathbf{e}_{N,N}\} \subseteq \Delta_\rho \subset \mathbb{R}^N.$$

The polytope \mathfrak{S}_ρ^N is obviously unimodularly equivalent to the closure of $\mathfrak{S}^N(1)$.

Theorem 7.2. *Let λ be a regular dominant integral weight and $k = \min\{\langle \lambda, \beta^\vee \rangle \mid \beta \in \Phi^+\}$. Fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$. Let Γ_λ be the associated finitely generated monoid and denote by $\Delta_\lambda \subset \mathbb{R}^N$ the associated Newton-Okounkov polytope. Then $k\mathfrak{S}_\rho^N \subset \Delta_\lambda$.*

Proof. Recall that $\lambda = k\rho + \mu$ for some dominant integral weight μ , which implies that $\text{es}(k\rho) \subset \text{es}(\lambda) = \Gamma_\lambda(1)$ by Theorem 5.4. Now Lemma 7.1 implies that $k\mathbf{e}_{1,N}, k\mathbf{e}_{2,N}, \dots, k\mathbf{e}_{N,N} \in \text{es}(k\rho)$ and hence:

$$\{0, k\mathbf{e}_{1,N}, k\mathbf{e}_{2,N}, \dots, k\mathbf{e}_{N,N}\} \subset \text{es}(\lambda) = \Gamma_\lambda(1) \subset \Delta_\lambda.$$

The convexity of Δ_λ implies that the convex hull of these points is contained in Δ_λ , and hence $k\mathfrak{S}_\rho^N \subseteq \Delta_\lambda$. \square

7.2. A special simplex in Δ_λ from cominuscule telescopes.

7.2.1. *A tower of Levi subalgebras.* Another approach to construct Newton-Okounkov bodies $\Delta_\lambda = \Delta_\lambda(\underline{\beta})$ containing particular simplices uses a tower of Levi subalgebras and is in this sense in the spirit of Thimm's trick [27]. We assume in the following that λ is a *regular dominant integral weight*. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be an enumeration of the simple roots of \mathfrak{g} , and $\varpi_1, \dots, \varpi_n$ be the associated fundamental weights. Let $\mathfrak{l}_j \subset \mathfrak{g}$ be the Levi subalgebra associated to the subset of simple roots $\{\alpha_1, \alpha_2, \dots, \alpha_j\}$. The enumeration induces an increasing sequence of Levi subalgebras

$$(7.1) \quad \mathfrak{l}_0 = \mathfrak{t} \subset \mathfrak{l}_1 \subset \mathfrak{l}_2 \subset \dots \subset \mathfrak{l}_n = \mathfrak{g}.$$

Set $\mathfrak{n}_j^- = \mathfrak{n}^- \cap \mathfrak{l}_j$, then we have an induced sequence of inclusions:

$$(7.2) \quad \mathfrak{n}_0^- = 0 \subset \mathfrak{n}_1^- \subset \mathfrak{n}_2^- \subset \dots \subset \mathfrak{n}_n^- = \mathfrak{n}^-.$$

Let $\Phi(\mathfrak{l}_j)^+$ be the set of positive roots of the Levi subalgebra \mathfrak{l}_j . For $j = 1, \dots, n$, set $\Phi_j^+ = \Phi(\mathfrak{l}_j)^+ \setminus \Phi(\mathfrak{l}_{j-1})^+$, then (7.2) induces a partition of the set of positive roots Φ^+ :

$$(7.3) \quad \Phi^+ = \Phi_1^+ \cup \Phi_2^+ \cup \dots \cup \Phi_n^+.$$

We fix now an enumeration of the set of positive roots which is compatible with the partition above. More precisely, there exist $N = i_1 > i_2 > \dots > i_{n-1} > i_n = 1$ such that for any $1 \leq j \leq N$:

$$(7.4) \quad \Phi_1^+ \cup \Phi_2^+ \cup \dots \cup \Phi_j^+ = \{\beta_{i_j}, \dots, \beta_N\}.$$

In the following, we take the PBW basis of the enveloping algebra $U(\mathfrak{n}^-)$ with respect to this enumeration. As in the sections before, from now on we fix as a total order the *opposite right lexicographic order* on the tuples and on the monomials.

7.2.2. *Levi subalgebras and essential monomials for $V(\varpi_j)$.* We will use the sequence in (7.1) to provide an inductive procedure to construct essential monomials. The fundamental weight ϖ_j can be viewed as a fundamental weight for \mathfrak{g} , as well as a fundamental weight for the Levi subalgebra \mathfrak{l}_j . We write $V(\varpi_j)$ for the irreducible \mathfrak{g} -representation and $\mathcal{V}(\varpi_j)$ for the irreducible \mathfrak{l}_j -representation. By fixing a highest weight vector v_{ϖ_j} in $V(\varpi_j)$, we may identify $\mathcal{V}(\varpi_j)$ with the cyclic \mathfrak{l}_j -submodule of $V(\varpi_j)$ generated by the highest weight vector v_{ϖ_j} .

The filtration of $U(\mathfrak{n}^-)$ defined in Section 5.1 also makes sense for the enveloping algebra $U(\mathfrak{n}_j^-)$. To be able to compare the filtrations of the algebras and the induced filtrations on the representations $V(\varpi_j)$ and $\mathcal{V}(\varpi_j)$, recall that there exists a number i_j (see (7.4)) such that $\Phi(\mathfrak{l}_j)^+ = \{\beta_{i_j}, \dots, \beta_N\}$. In the following we take the PBW-basis of $U(\mathfrak{n}_j^-)$ with respect to this enumeration, which are monomials of the form

$F_{\beta_{i_j}}^{m_{i_j}} \dots F_{\beta_N}^{m_N}$. We can thus identify the enveloping algebra $U(\mathfrak{n}_j^-)$ with the linear span of all ordered monomials $F^{\underline{\mathbf{m}}}$ in $U(\mathfrak{n}^-)$ such that

$$(7.5) \quad m_1 = m_2 = \dots = m_{i_j-1} = 0.$$

If $\underline{\mathbf{m}}$ is as in (7.5), the monomial $F^{\underline{\mathbf{m}}}$ is an element of both $U(\mathfrak{n}_j^-)$ and $U(\mathfrak{n}^-)$, giving two notions of being essential: $\underline{\mathbf{m}}$ can be essential for either the \mathfrak{l}_j -representation $\mathcal{V}(\varpi_j)$, or the \mathfrak{g} -representation $V(\varpi_j)$.

Lemma 7.3. *If $\underline{\mathbf{m}}$ is as in (7.5), then $\underline{\mathbf{m}}$ is essential for the \mathfrak{l}_j -representation $\mathcal{V}(\varpi_j)$ if and only if it is so for the \mathfrak{g} -representation $V(\varpi_j)$.*

Proof. Suppose that the tuples $\underline{\mathbf{a}}^1, \dots, \underline{\mathbf{a}}^r \in \mathbb{N}^N$ are essential for ϖ_j , satisfying $\underline{\mathbf{a}}^k < \underline{\mathbf{m}}$ for $k = 1, \dots, r$ and

$$(7.6) \quad F^{\underline{\mathbf{m}}} v_{\varpi_j} = \sum_{k=1}^r c_k F^{\underline{\mathbf{a}}^k} v_{\varpi_j}.$$

All positive roots such that the root vector F_γ occurs in $F^{\underline{\mathbf{m}}}$ are elements in $\Phi(\mathfrak{l}_j)^+$ by assumption, and hence these roots are linear combinations of the simple roots $\alpha_1, \dots, \alpha_j$. For weight reasons, all positive roots γ such that the root vector F_γ occurs in one of the monomials on the right hand side in (7.6) must also be linear combinations of the simple roots $\alpha_1, \dots, \alpha_j$. Hence the monomials occurring on the right hand side are elements of the enveloping algebra of \mathfrak{l}_j . This implies that in this special situation being essential for $\mathcal{V}(\varpi_j)$ is equivalent to being essential for $V(\varpi_j)$. \square

7.2.3. Some special essential monomials for cominuscule weights.

Lemma 7.4. *If ϖ_j is a cominuscule weight for \mathfrak{l}_j , then all root vectors F_β , $\beta \in \Phi_j^+$, are essential monomials for the \mathfrak{g} -representation $V(\varpi_j)$.*

Proof. By Lemma 7.3 it is sufficient to prove that F_β is an essential monomial for the \mathfrak{l}_j -representation $\mathcal{V}(\varpi_j)$. Let $\underline{\mathbf{m}}_\beta \in \mathbb{N}^N$ be such that $F_\beta = F^{\underline{\mathbf{m}}_\beta}$. Suppose there exist monomials $F^{\underline{\mathbf{a}}^1}, \dots, F^{\underline{\mathbf{a}}^r} \in U(\mathfrak{n}_j^-)$ such that $\underline{\mathbf{a}}^k < \underline{\mathbf{m}}_\beta$ for $k = 1, \dots, r$, and

$$(7.7) \quad F^{\underline{\mathbf{m}}_\beta} v_{\varpi_j} = \sum_{k=1}^r c_k F^{\underline{\mathbf{a}}^k} v_{\varpi_j}.$$

For a monomial $F^{\underline{\mathbf{a}}^k}$ with nonzero coefficient c_k and a positive root $\gamma_i \in \Phi^+(\mathfrak{l}_j)$, let a_i be the exponent of F_{γ_i} in the monomial. Comparing the weights on both sides of (7.7), the sum $\sum_{\ell=i_j}^N a_\ell \gamma_\ell$ is equal to β . Since ϖ_j is assumed to be cominuscule, the coefficient of α_j in the expression of β as a sum of simple roots is equal to one. It follows, that at least for one of the roots one has that $a_i > 0$ but $\gamma_i \notin \Phi_j^+$. This implies $\gamma_i \in \Phi^+(\mathfrak{l}_{j-1})$, and, because of the fixed order in which the monomials are written, we have $F^{\underline{\mathbf{a}}^k} v_{\varpi_j} = 0$. As a consequence, a linear dependence relation as in (7.7) is not possible. \square

Theorem 7.5. *Let λ be a regular dominant integral weight and let $k = \min\{\langle \lambda, \beta^\vee \rangle \mid \beta \in \Phi^+\}$. If \mathfrak{g} is not of type G_2, F_4 or E_8 , then there exists an enumeration of the positive roots such that Γ_λ is finitely generated and $\mathfrak{S}^N(k) \subset \Delta_\lambda$.*

Proof. By [7], $\Gamma_\lambda = \text{Es}(\lambda)$ is finitely generated if, for example, one can show that there exists a reduced decomposition of w_0 such that the induced ordering has the properties demanded in (7.4).

If \mathfrak{g} is not of type G_2, F_4 or E_8 , then there exists an enumeration $\{\alpha_1, \dots, \alpha_n\}$ of the simple roots such that for all $j = 1, \dots, n$, the fundamental weight ϖ_j is cominuscule for the Levi subalgebra \mathfrak{l}_j , i.e., $\langle \varpi_j, \theta_j^\vee \rangle = 1$ for the highest root θ_j in \mathfrak{l}_j . For \mathfrak{g} of type A_n , any enumeration of the simple roots has this property. For \mathfrak{g} of type C_n or D_n , one takes the standard enumeration as in [4]. For \mathfrak{g} of type B_n take α_1 to be the only short root among the simple roots, then add successively the only simple root which is connected in the Dynkin diagram to the already chosen ones. For \mathfrak{g} of type E_6 take first the enumeration of the simple roots as in the D_5 case above, and then add the only missing simple root as the sixth one. For \mathfrak{g} of type E_7 take first the enumeration of the simple roots as in the E_6 case above, and then add the only missing simple root as the seventh one.

To get an enumeration of the positive roots induced by a reduced decomposition and also satisfying (7.4), let w_0 be the longest word in the Weyl group W of G , w_0^j be the longest word in the Weyl group of \mathfrak{l}_j , and

$$\tau_j \equiv w_0^j \pmod{W_{j-1}}$$

be a minimal representative in W_j of the class of w_0^j in $W_{j-1} \backslash W_j$. Note that the block decomposition $w_0^j = \tau_1 \tau_2 \cdots \tau_j$ is such that the lengths add up, i.e., $\ell(w_0^j) = \sum_{s=1}^j \ell(\tau_s)$ for all $j = 1, \dots, n$. We fix a reduced decomposition of the longest word w_0 which is compatible with this block decomposition, i.e., the decomposition of the form:

$$(7.8) \quad w_0 = \underbrace{s_1}_{\tau_1} \underbrace{s_2 s_1 \cdots}_{\tau_2} \cdots \underbrace{s_{j_1} \cdots s_{j_r}}_{\tau_j} \cdots \underbrace{s_{n_1} \cdots s_{n_t}}_{\tau_n}.$$

It is now easy to see that the enumeration of the elements of Φ^+ induced by this reduced decomposition (see (5.2), here $L = T$) has the properties described in (7.4).

Lemma 7.4 implies that with respect to this enumeration, every root vector is essential for some fundamental representation. Now the same arguments as in the proof of Theorem 7.2 imply that for any $i = 1, \dots, N$, \mathbf{e}_i is essential for $V(\rho)$. By the same arguments as above, k is maximal such that $\lambda = k\rho + \mu$ for some dominant weight μ , so $\text{es}(k\rho) \subset \text{es}(\lambda)$ and hence

$$\{0, k\mathbf{e}_1, k\mathbf{e}_2, \dots, k\mathbf{e}_N\} \subset \Delta_\lambda.$$

Since Δ_λ is convex, it follows that $\mathfrak{S}^N(k) \subset \Delta_\lambda$. \square

Remark 7.6. Recall that by our construction the Newton-Okounkov body is contained in the positive octant, thus in this case $\mathfrak{S}^N(k)$ is the intersection of Δ_λ with an affine half-space. Remark 3.3 implies hence that the supremum appearing in the definition of Gromov width is attained: there exists a symplectic embedding of a ball of capacity k .

We give several examples on the construction in Theorem 7.5.

Example 7.7. We fix the following notations: for $1 \leq i \leq j \leq n$, $\alpha_{i,j} = \alpha_i + \dots + \alpha_j$; when \mathfrak{g} is of type B_n , for $1 \leq i < j \leq n$, $\alpha_{i,\bar{j}} = \alpha_i + \dots + \alpha_n + \alpha_n + \dots + \alpha_j$; when \mathfrak{g} is of type C_n , for $1 \leq i \leq j \leq n$, $\alpha_{i,\bar{j}} = \alpha_i + \dots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \dots + \alpha_j$.

- (1) For \mathfrak{g} of type A_n , by construction, we may consider the enumeration of positive roots arising from the following inclusions of Levi subalgebras:

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \dots \subset \mathfrak{sl}_n \subset \mathfrak{sl}_{n+1}.$$

For example, when $n = 3$, the enumeration can be chosen as

$$(\alpha_{3,3}, \alpha_{2,3}, \alpha_{1,3}, \underbrace{\alpha_{2,2}, \alpha_{1,2}, \alpha_{1,1}}_{A_2}).$$

- (2) For \mathfrak{g} of type B_n , we consider the enumeration of positive roots arising from the following inclusions of Levi subalgebras:

$$\mathfrak{sl}_2 \subset \mathfrak{so}_5 \subset \mathfrak{so}_7 \subset \dots \subset \mathfrak{so}_{2n-1} \subset \mathfrak{so}_{2n+1}.$$

For instance, when $n = 3$, the enumeration can be chosen as

$$(\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{1,\bar{3}}, \alpha_{1,\bar{2}}, \underbrace{\alpha_{2,2}, \alpha_{2,3}, \alpha_{2,\bar{3}}, \alpha_{3,3}}_{B_2}).$$

- (3) For \mathfrak{g} of type C_n , we consider the enumeration of positive roots arising from the following inclusions of Levi subalgebras:

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \dots \subset \mathfrak{sl}_n \subset \mathfrak{sp}_{2n}.$$

For example, when $n = 3$, the enumeration can be chosen as

$$(\alpha_{3,3}, \alpha_{2,3}, \alpha_{2,\bar{2}}, \alpha_{1,3}, \alpha_{1,\bar{2}}, \alpha_{1,\bar{1}}, \underbrace{\alpha_{2,2}, \alpha_{1,2}, \alpha_{1,1}}_{A_2}).$$

- (4) For \mathfrak{g} of type D_n , there are different ways to obtain enumerations of positive roots from inclusions of Levi subalgebras. For example,

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \mathfrak{sl}_4 \subset \mathfrak{so}_8 \subset \dots \subset \mathfrak{so}_{2n-2} \subset \mathfrak{so}_{2n} \quad \text{or} \quad \mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \dots \subset \mathfrak{sl}_n \subset \mathfrak{so}_{2n}.$$

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MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, COLOGNE, GERMANY

E-mail address: xinfang.math@gmail.com

E-mail address: peter.littelmann@math.uni-koeln.de

E-mail address: pabiniak@math.uni-koeln.de